ON THE THEORY OF PLANE STRAIN FOR A STRAIN-HARDENING PLASTIC MATERIAL

(K TEORII PLOSKOI DEFORMATSII UPROCHNIAIUSHCHEGOSIA PLASTICHESKOGO MATERIALA)

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The simplest version of the theory of plasticity is the theory of an incompressible ideal isotropic rigid plastic body. In this case the plasticity condition is fixed, depending, generally speaking, on the second or third invariant of the stress deviator tensor.

In the plane of the deviator of the principal stresses the plasticity condition is interpreted as a certain curve called the curve of plasticity. Well known generalizations of the theory of ideal plasticity consist in assumptions of the changes in form of the flow curve in dependence to the deformed state [1-6].

Changes in the flow limit during deformation characterize the strain-hardening materials, in which if the body remains isotropic under strain, the process is called isotropic strain-hardening. With isotropic strain-hardening the plasticity condition may depend upon the second or third invariants of the deviators of the stress and strain tensors. In this case the plasticity curve remains symmetrical relative to the axes of the principal stresses.

If the flow limits for the different stresses do not coincide, then the material is anisotropic. One of the simplest versions of the theory of anisotropic strain-hardening was first proposed by Prager [1], later investigated in [4-7]. The flow curve is shifted as a whole, and the plasticity condition depends on mixed invariants of the deviators of the stress and strain tensors. We note the mechanical interpretation of the nature of anisotropic strain-hardening proposed in [5], which clarifies the role of microstresses in the framework of phenomenological theory.

The relations of the theory of anisotropic strain-hardening are considered here, including the special case obtained from the theory of isotropic strain-hardening as well as Prager's theory of anisotropic strain-hardening. The exposition is for the case of plane strain.

The plasticity condition is assumed in the form

$$f(\Sigma_2, \Sigma_3, \Gamma_2, \Gamma_3, T_2, T_3) = 0 \tag{1}$$

Here Σ_2, Σ_3 denote the second and third invariants of the deviator of the stress tensor σ_{ij}; Γ_2, Γ_3 denote the second and third invariants of the deviator of the strain tensor ϵ_{ij}; and T_2, T_3 denote the second and third invariants of the deviator of the tensor $(\sigma_{ij} - c\epsilon_{ij})$, where $c = c(\Gamma_2, \Gamma_3)$.

In accordance with this the associated law of plastic flow is

$$d\varepsilon_{ij} = d\lambda \frac{\partial f}{\partial \sigma_{ij}} \tag{2}$$

Since

$$\Sigma_2 = \tfrac{1}{6}\left[(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau^2{}_{xy} + \tau_{yz}{}^2 + \tau_{zx}{}^2)\right]$$
$$\Sigma_3 = s_x s_y s_z + 2\tau_{xy}\tau_{yz}\tau_{zx} - s_x\tau_{yz}{}^2 - s_y\tau_{zx}{}^2 - s_z\tau_{xy}{}^2 \tag{3}$$

where $s_i = \sigma_i - \sigma$, $\sigma = 1/3(\sigma_x + \sigma_y + \sigma_z)$, and the remaining invariants are written analogously, then

$$\begin{aligned} d\varepsilon_x = d\lambda \Big\{ & \frac{1}{3}\frac{\partial f}{\partial \Sigma_2}(2\sigma_x - \sigma_y - \sigma_z) + \frac{\partial f}{\partial \Sigma_3}\left(s_y s_z - \tau_{yz}^2 + \frac{1}{3}\Sigma_2\right) + \\ & + \frac{1}{3}\frac{\partial f}{\partial T_2}[2(\sigma_x - c\varepsilon_x) - (\sigma_y - c\varepsilon_y) - (\sigma_z - c\varepsilon_z)] + \\ & + \frac{\partial f}{\partial T_3}\left[(s_y - c\varepsilon_y)(s_z - c\varepsilon_z) - (\tau_{yz} - c\varepsilon_{yz})^2 + \frac{1}{3}T_2\right]\Big\}, \ldots \end{aligned} \tag{4}$$

$$\begin{aligned} d\varepsilon_{xy} = d\lambda \Big\{ & \frac{\partial f}{\partial \Sigma_2}\tau_{xy} + \frac{\partial f}{\partial \Sigma_3}(\tau_{yz}\tau_{zx} - s_z\tau_{xy}) + \frac{\partial f}{\partial T_2}(\tau_{xy} - c\varepsilon_{xy}) + \\ & + \frac{\partial f}{\partial T_3}[(\tau_{xz} - c\varepsilon_{xz})(\tau_{yz} - c\varepsilon_{yz}) - (s_z - c\varepsilon_z)(\tau_{xy} - c\varepsilon_{xy})]\Big\} = 0, \ldots \end{aligned}$$

The remaining expressions are obtained by successive permutation of indices. It is evident that the compressibility condition $\epsilon_x + \epsilon_y + \epsilon_z = 0$ holds.

We consider the case of plane strain. We set

$$\begin{aligned} &\varepsilon_z = \varepsilon_{xz} = \varepsilon_{yz} = \tau_{xz} = \tau_{yz} = 0, \quad \sigma_x = \sigma_x(x, y), \quad \sigma_y = \sigma_y(x, y) \\ &\tau_{xy} = \tau_{xy}(x, y), \ \ \varepsilon_x = \varepsilon_x(x, y), \quad \varepsilon_y = \varepsilon_y(x, y), \quad \varepsilon_{xy} = \varepsilon_{xy}(x, y) \end{aligned} \tag{5}$$

From (4) for $\epsilon_z = 0$ we have

$$\frac{1}{3}\frac{\partial f}{\partial \Sigma_2}(2\sigma_z - \sigma_x - \sigma_y) + \frac{\partial f}{\partial \Sigma_3}\left(s_x s_y - \tau_{xy}^2 + \frac{1}{3}\Sigma_2\right) +$$
$$+\frac{1}{3}\frac{\partial f}{\partial T_2}(2\sigma_z - \sigma_x - \sigma_y) + \frac{\partial f}{\partial T_3}\left[(s_x - c\varepsilon_x)(s_y - c\varepsilon_y) - (\tau_{xy} - c\varepsilon_{xy})^2 + \frac{1}{3}T_2\right] = 0 \quad (6)$$

The conditions $\epsilon_{xz} = \epsilon_{yz} = \tau_{xz} = \tau_{yz} = 0$ are satisfied identically. It is evident that under the conditions (5) and for

$$\sigma_z = \frac{1}{2}(\sigma_x + \sigma_y) \quad (7)$$

the relation

$$\Sigma_3 = T_3 = 0$$

holds.

Condition (6) will be fulfilled if (7) holds, and also

$$\frac{\partial f}{\partial \Sigma_3} = \frac{\partial f}{\partial T_3} = 0 \qquad \text{for } \Sigma_3 = T_3 = 0 \quad (8)$$

We shall assume in the sequel conditions that (7) and (8) are satisfied; the relations for the case of plane strain are written in the form

$$\begin{aligned} d\varepsilon_x &= \frac{1}{2}d\lambda\left\{\frac{\partial f}{\partial \Sigma_2}(\sigma_x - \sigma_y) + \frac{\partial f}{\partial T_2}[(\sigma_x - c\varepsilon_x) - (\sigma_y - c\varepsilon_y)]\right\} \\ d\varepsilon_y &= \frac{1}{2}d\lambda\left\{\frac{\partial f}{\partial \Sigma_2}(\sigma_y - \sigma_x) + \frac{\partial f}{\partial T_2}[(\sigma_y - c\varepsilon_y) - (\sigma_x - c\varepsilon_x)]\right\} \\ d\varepsilon_{xy} &= d\lambda\left[\frac{\partial f}{\partial \Sigma_2}\tau_{xy} + \frac{\partial f}{\partial T_2}(\tau_{xy} - c\varepsilon_{xy})\right] \end{aligned} \quad (9)$$

The plasticity condition may be presented in the form

$$f(\Sigma_2^*, \Gamma_2^*, T_2^*) = 0 \quad (10)$$

where

$$\Sigma_2^* = \frac{1}{4}(\sigma_x - \sigma_y)^2 + \tau_{xy}^2, \qquad \Gamma_2^* = \frac{1}{4}(\varepsilon_x - \varepsilon_y)^2 + \varepsilon_{xy}^2$$
$$T_2^* = \frac{1}{4}[(\sigma_x - c\varepsilon_x) - (\sigma_y - c\varepsilon_y)]^2 + (\tau_{xy} - c\varepsilon_{xy})^2$$

The star above and the index two are omitted in the following.

We shall consider only the case of the plasticity condition in the form

$$T = T(\Gamma)$$

or

$$[(\sigma_x - c\varepsilon_x) - (\sigma_y - c\varepsilon_y)]^2 + 4(\tau_{xy} - c\varepsilon_{xy})^2 = 4k^2(\Gamma) \quad (11)$$

The relations (9) are written in the form

$$\frac{d\varepsilon_x}{(\sigma_x - c\varepsilon_x) - (\sigma_y - c\varepsilon_y)} = \frac{d\varepsilon_y}{(\sigma_y - c\varepsilon_y) - (\sigma_x - c\varepsilon_x)} = \frac{d\varepsilon_{xy}}{2(\tau_{xy} - c\varepsilon_{xy})} \tag{12}$$

The condition (11) is satisfied by the assumptions

$$\sigma_x = \sigma + c\varepsilon_x + k\cos 2\theta, \qquad \sigma_y = \sigma + c\varepsilon_y - k\cos 2\theta, \quad \tau_{xy} = c\varepsilon_{xy} + k\sin 2\theta \tag{13}$$

We now denote $\epsilon_x = -\epsilon_y = \epsilon$, $\epsilon_{xy} = \gamma$. Then it is evident that $\Gamma = \epsilon^2 + \gamma^2$.

Expressions (13) are substituted in the equations of equilibrium

$$\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} = 0, \qquad \frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_y}{\partial y} = 0 \tag{14}$$

and we obtain

$$\begin{aligned}
&\frac{\partial\sigma}{\partial x} - 2k\sin 2\theta\frac{\partial\theta}{\partial x} + 2k\cos 2\theta\frac{\partial\theta}{\partial y} + c\frac{\partial\varepsilon}{\partial x} + 2\left(\frac{dk}{d\Gamma}\cos 2\theta + \frac{dc}{d\Gamma}\varepsilon\right)\times\\
&\times\left(\varepsilon\frac{\partial\varepsilon}{\partial x} + \gamma\frac{\partial\gamma}{\partial x}\right) + c\frac{\partial\gamma}{\partial y} + 2\left(\frac{dk}{d\Gamma}\sin 2\theta + \frac{dc}{d\Gamma}\gamma\right)\left(\varepsilon\frac{\partial\varepsilon}{\partial y} + \gamma\frac{\partial\gamma}{\partial y}\right) = 0\\
&\frac{\partial\sigma}{\partial y} + 2k\cos 2\theta\frac{\partial\theta}{\partial x} + 2k\sin 2\theta\frac{\partial\theta}{\partial y} - c\frac{\partial\varepsilon}{\partial y} + 2\left(\frac{dk}{d\Gamma}\sin 2\theta + \frac{dc}{d\Gamma}\gamma\right)\times\\
&\times\left(\varepsilon\frac{\partial\varepsilon}{\partial x} + \gamma\frac{\partial\gamma}{\partial x}\right) + c\frac{\partial\gamma}{\partial x} - 2\left(\frac{dk}{d\Gamma}\cos 2\theta + \frac{dc}{d\Gamma}\varepsilon\right)\left(\varepsilon\frac{\partial\varepsilon}{\partial y} + \gamma\frac{\partial\gamma}{\partial y}\right) = 0
\end{aligned} \tag{15}$$

Relations (12) are of the form

$$d\varepsilon\sin 2\theta - d\gamma\cos 2\theta = 0 \tag{16}$$

Considering also the compatibility conditions

$$\frac{\partial\omega}{\partial x} - \frac{\partial\varepsilon}{\partial y} + \frac{\partial\gamma}{\partial x} = 0, \qquad \frac{\partial\omega}{\partial y} - \frac{\partial\varepsilon}{\partial x} - \frac{\partial\gamma}{\partial y} = 0 \tag{17}$$

we obtain a system of five quasilinear equations (15), (16), (17) in the five unknowns σ, θ, ϵ, γ, ω.

Let us investigate the type of this system. Upon denoting $\psi(x, y)$ as the equation of the characteristic surface we set up the characteristic determinant. It will have the form

$$\begin{vmatrix}
\psi & -2k(\psi_x\sin 2\theta - \psi_y\cos 2\theta) & \dots & \dots & \dots\\
\psi_y & 2k(\psi_x\cos 2\theta + \psi_y\sin 2\theta) & \dots & \dots & \dots\\
0 & 0 & d\psi\sin 2\theta & -d\psi\cos 2\theta & 0\\
0 & 0 & -\psi_y & \psi_x & \psi_x\\
0 & 0 & -\psi_x & -\psi_y & \psi_y
\end{vmatrix} = 0$$

where $\psi_x = \partial\psi/\partial x$, $\psi_y = \partial\psi/\partial y$.

It is evident that the characteristic determinant is equal to the product of two diagonals of the second and third orders. Terms in the first and second parentheses show no effect on the determinant. These terms are zero for ideal plasticity. Assuming $d\psi \neq 0$, $k \neq 0$, we obtain

$$\psi_x^2 \cos 2\theta + 2\psi_x\psi_y \sin 2\theta \cos 2\theta - \psi_y^2 \cos 2\theta = 0 \tag{18}$$

The system is thus shown to be of the hyperbolic type, and the characteristics are mutually orthogonal. The nature of the strain-hardening appears in expressions which are generalized Hencky relations. Upon employing a change of variables

$$\begin{aligned} d\xi &= dy \cos\left(\theta - \tfrac{1}{4}\pi\right) - dx \sin\left(\theta - \tfrac{1}{4}\pi\right) \\ d\eta &= dy \cos\left(\theta + \tfrac{1}{4}\pi\right) - dx \sin\left(\theta + \tfrac{1}{4}\pi\right) \end{aligned} \tag{19}$$

we obtain, from (15), along the characteristics

$$\begin{gathered} \frac{\partial\sigma}{\partial\xi} + 2k\frac{\partial\theta}{\partial\xi} - c\left[\frac{\partial\varepsilon}{\partial\eta}\cos 2\theta + \frac{\partial\gamma}{\partial\eta}\sin 2\theta\right] + \\ + \frac{dc}{d\Gamma}\left[(\gamma\cos 2\theta - \varepsilon\sin 2\theta)\frac{\partial\Gamma}{\partial\xi} - \left(\gamma\sin 2\theta + \varepsilon\cos 2\theta + \frac{dk}{d\Gamma}\right)\frac{\partial\Gamma}{\partial\eta}\right] = 0 \\ -\frac{\partial\sigma}{\partial\eta} + 2k\frac{\partial\theta}{\partial\eta} + c\left[\frac{\partial\varepsilon}{\partial\xi}\cos 2\theta + \frac{\partial\gamma}{\partial\xi}\sin 2\theta\right] + \\ + \frac{dc}{d\Gamma}\left[\left(\gamma\sin 2\theta + \varepsilon\cos 2\theta + \frac{dk}{d\Gamma}\right)\frac{\partial\Gamma}{\partial\xi} + (\gamma\cos 2\theta - \varepsilon\sin 2\theta)\frac{\partial\Gamma}{\partial\eta}\right] = 0 \end{gathered} \tag{20}$$

We note certain special cases: for $c = 0$ we have the relations for isotropic strain-hardening; for k = const we have the relations given in [7]; for $c = 0$ and k = const we have the Hencky relations.

Equation (16) is transformable into the Geiringer relation, thus confirming the absence of elongations along the characteristics

$$dU - Vd\theta = 0, \qquad dV + U\,d\theta = 0 \tag{21}$$

where U and V are displacement velocities along the characteristics.

We introduce the angles μ and ν determining the direction of the axes of the stress and strain tensors in the xy plane

$$\tan 2\mu = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}, \qquad \tan 2\nu = \frac{2\varepsilon_{xy}}{\varepsilon_x - \varepsilon_y}$$

We have

$$\begin{aligned} \sigma_x &= \sigma + \Sigma\cos 2\mu, & \varepsilon_x &= \Gamma\cos 2\nu \\ \sigma_y &= \sigma - \Sigma\cos 2\mu, & \varepsilon_y &= -\Gamma\cos 2\nu \\ \tau_{xy} &= \Sigma\sin 2\mu, & \varepsilon_{xy} &= \Gamma\sin 2\nu \end{aligned} \tag{22}$$

We obtain from (13) and (22)

$$\Sigma^2 + p^2 - 2\,\Sigma p \cos^2(\mu - \nu) = k^2, \qquad p = \Gamma c\,(\Gamma)$$

It is evident that the relation $\Sigma = \Sigma(\Gamma)$ holds only if the directions of the principal axes of the stress and strain tensors coincide, $\mu = \nu$; otherwise the relation describes the stress history. Consideration of the theory of torsion presents no difficulty; in this case the third invariant is also zero. The three-dimensional problem may be considered following the procedure indicated in [7].

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